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# Propagation of Thermoelastic Waves in Micropolar Mixture of Porous Media

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**Abstract** The theory of coupled thermoelasticity for a micropolar mixture of porous media (Eringen AC, J Appl Phys 94:909, 2003) is generalized in the context of Lord and Shulman and Green and Lindsay theories of generalized thermoelasticity. The governing equations of generalized thermoelasticity of a micropolar mixture of porous media are solved to show the existence of three coupled longitudinal displacement waves, two coupled longitudinal microrotational waves, and six coupled transverse waves, which attenuate and are dispersive in nature.

**Keywords** Attenuation · Dispersion · Micropolar mixture · Thermal relaxation times · Wave propagation

## 1 Introduction

Biot [1] formulated the theory of coupled thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. The heat equations for both coupled and uncoupled theories of the diffusion type, predicting infinite speeds of propagation for heat waves, are contrary to physical observations. Hetnarski and Ignaczack [2] examined five generalizations to the coupled theory and obtained a number of important analytical results. The first generalized theory of thermoelasticity is due to Lord and Shulman [3] who introduced the theory of generalized thermoelasticity with one relaxation time by postulating a new law of heat conduction to replace the classical Fourier law. This new law contains the heat flux vector as well as its time derivative. It contains also a new constant that acts as a relaxation time. The heat equation of this theory is of the wave-type, ensuring finite

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speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equation of motion and the constitutive relations remain the same as those for the coupled and uncoupled theories. The second generalization to the coupled theory is known as the generalized theory with two relaxation times. Muller [4] introduced the theory of generalized thermoelasticity with two relaxation times. A more explicit version was then introduced by Green and Laws [5], Green and Lindsay [6], and independently by Suhubi [7]. In this theory the temperature rates are considered among the constitutive variables. This theory also predicted finite speeds of propagation as in the Lord–Shulman theory. It differs from the latter in that Fourier's law of heat conduction is not violated if the body under consideration has a center of symmetry. A unified treatment of both Lord–Shulman and Green–Lindsay theories was presented by Ignaczak and Ostoja-Starzewski [8].

All materials, whether natural or synthetic, possess microstructures. The classical theory of elasticity is inadequate to represent the behavior of materials possessing microstructure. Eringen [9] developed a linear theory of micropolar elasticity which takes into account the intrinsic rotations of grains. Soils, rocks, granular materials, sand, underground water mixtures, etc. are typical media which may be modeled more realistically by means of the micropolar theory of elasticity. The micropolar theory was extended to include thermal effects by Nowacki [10–13], Nowacki and Olszak [14], Eringen [15], Tauchert et al. [16], and Tauchert [17]. Boschi and Iesan [18] and Dost and Tabarrok [19] presented generalized theories of micropolar thermoelasticity. Kumar and Singh [20] studied wave propagation in a micropolar generalized thermoelastic body with stretch.

Twiss and Eringen [21,22] presented the first extensive theory of mixtures for micromorphic solids, which is a more general micropolar mixture theory and where 12 degrees of freedom are required for deformation of material points. Recently, Eringen [23] developed the theory of a micropolar mixture of porous media (non-reacting mixture of a micropolar elastic solid and a micropolar viscous fluid at a single temperature) to include rotational degrees of freedom. In this theory, the deformation of material points requires six degrees of freedom. This theory is intended for applications of soil with grains, porous solids with grains and tortuosity, sand, water or oil, etc. Wave propagation in media represented by this theory may provide useful information for its applications in consolidation and oil exploration problems. In the present article, the coupled thermoelasticity of a micropolar mixture of porous media given by Eringen [23] is generalized in the context of Lord–Shulman (L-S) and Green–Lindsay (G-L) theories. The governing equations of generalized thermoelasticity of micropolar mixtures of porous media are solved to show the existence of thermoelastic waves.

## 2 Equations of Motion

A binary mixture of a micropolar elastic solid and a micropolar fluid is considered at the same temperature where no chemical reaction takes place between the two micro-isotropic species of the mixture. Following Eringen [23], Lord and Shulman [3], and Green and Lindsay [6], the equations of motion in a linear, isotropic thermoelastic mixture of a micropolar elastic solid and a micropolar viscous fluid in the absence of external loads, can be written as

$$
\begin{aligned}
&(\lambda^{\mathrm{S}}+2\mu^{\mathrm{S}}+\kappa^{\mathrm{S}})\nabla(\nabla\cdot\mathbf{u}^{\mathrm{S}})-(\mu^{\mathrm{S}}+\kappa^{\mathrm{S}})\nabla\times(\nabla\times\mathbf{u}^{\mathrm{S}})\\
&\quad+\kappa^{\mathrm{S}}\nabla\times\phi^{\mathrm{S}}-\xi(\dot{\mathbf{u}}^{\mathrm{S}}-\dot{\mathbf{u}}^{\mathrm{F}})-\left(\frac{s}{T_0}+\beta_0\right)\nabla(T+t_1\dot{T})=\rho^{\mathrm{S}}\ddot{\mathbf{u}}^{\mathrm{S}}, \qquad (1)\\
&(\alpha^{\mathrm{S}}+\beta^{\mathrm{S}}+\gamma^{\mathrm{S}})\nabla(\nabla\cdot\phi^{\mathrm{S}})-\gamma^{\mathrm{S}}\nabla\times(\nabla\times\phi^{\mathrm{S}})\\
&\quad+\kappa^{\mathrm{S}}(\nabla\times\mathbf{u}^{\mathrm{S}}-2\phi^{\mathrm{S}})-\bar{\omega}(\dot{\phi}^{\mathrm{S}}-\dot{\phi}^{\mathrm{F}})=\rho^{\mathrm{S}}j^{\mathrm{S}}\ddot{\phi}^{\mathrm{S}}, \qquad (2)\\
&(\lambda^{\mathrm{F}}+2\mu^{\mathrm{F}}+\kappa^{\mathrm{F}})\nabla(\nabla\cdot\dot{\mathbf{u}}^{\mathrm{F}})-(\mu^{\mathrm{F}}+\kappa^{\mathrm{F}})\nabla\times(\nabla\times\dot{\mathbf{u}}^{\mathrm{F}})\\
&\quad+\kappa^{\mathrm{F}}\nabla\times\dot{\phi}^{\mathrm{F}}+\xi(\dot{\mathbf{u}}^{\mathrm{S}}-\dot{\mathbf{u}}^{\mathrm{F}})+\left(\frac{s}{T_0}+\beta_0\right)\nabla(T+t_1\dot{T})=\rho^{\mathrm{F}}\ddot{\mathbf{u}}^{\mathrm{F}}, \qquad (3)\\
&(\alpha^{\mathrm{F}}+\beta^{\mathrm{F}}+\gamma^{\mathrm{F}})\nabla(\nabla\cdot\dot{\phi}^{\mathrm{F}})-\gamma^{\mathrm{F}}\nabla\times(\nabla\times\dot{\phi}^{\mathrm{F}})\\
&\quad+\kappa^{\mathrm{F}}(\nabla\times\dot{\mathbf{u}}^{\mathrm{F}}-2\dot{\phi}^{\mathrm{F}})+\bar{\omega}(\dot{\phi}^{\mathrm{S}}-\dot{\phi}^{\mathrm{F}})=\rho^{\mathrm{F}}j^{\mathrm{F}}\ddot{\phi}^{\mathrm{F}}, \qquad (4)\\
&\nabla\cdot\left[s(\dot{\mathbf{u}}^{\mathrm{S}}-\dot{\mathbf{u}}^{\mathrm{F}})+\frac{K}{T_0}\nabla T\right]\\
&\quad-\rho\left[\beta_0\nabla\cdot(\dot{\mathbf{u}}^{\mathrm{S}}+t_0\Delta\ddot{\mathbf{u}}^{\mathrm{S}})+\rho^{\mathrm{F}}\delta\nabla\cdot(\dot{\mathbf{u}}^{\mathrm{F}}+t_0\Delta\ddot{\mathbf{u}}^{\mathrm{F}})\right]=\rho C_0(\dot{T}+t_0\ddot{T}). \qquad (5)
\end{aligned}
$$

where $\mathbf{u}^{\mathrm{S}}, \mathbf{u}^{\mathrm{F}}, \phi^{\mathrm{S}}, \phi^{\mathrm{F}}$, and $T$ are independent variables. The coefficients $\lambda^{\mathrm{S}}, \mu^{\mathrm{S}}, \kappa^{\mathrm{S}}, \alpha^{\mathrm{S}}, \beta^{\mathrm{S}}, \gamma^{\mathrm{S}}, \beta_0$ are the micropolar thermoelastic constants for isotropic micropolar elastic solids, $\lambda^{\mathrm{F}}, \mu^{\mathrm{F}}, \kappa^{\mathrm{F}}, \alpha^{\mathrm{F}}, \beta^{\mathrm{F}}, \gamma^{\mathrm{F}}$ are the micropolar fluid viscosities, $K$ is the classical Fourier constant, $s$ is the heat generation due to the velocity difference, $\xi$ is the momentum generation coefficient due to the velocity difference, $\bar{\omega}$ is the momentum generation coefficient due to the difference in gyrations, $C_0$ is the specific heat at constant deformation, $T_0$ is ambient temperature, $\delta = -(\frac{\partial\psi}{\partial\rho^{\mathrm{F}}})_0$, $\psi$ is Helmholtz's free energy function, $j^{\mathrm{S}}$ and $j^{\mathrm{F}}$ are the micro-inertia of the solid and fluid, respectively, $\rho, \rho^{\mathrm{S}}$, and $\rho^{\mathrm{F}}$ are the densities of the mixture, solid, and fluid, respectively, the superposed dot indicates the temporal derivative, and other symbols have their usual meanings.

The use of the symbol $\Delta$ in Eq. 5 makes these fundamental equations possible for the two different theories of the generalized thermoelasticity. For the L-S (Lord–Shulman) theory, $t_1 = 0,\ \Delta = 1$, and for the G-L (Green–Lindsay) theory, $t_1 > 0$ and $\Delta = 0$. The thermal relaxations $t_0$ and $t_1$ satisfy the inequality $t_1 > t_0 > 0$ for the G-L theory only.

By introducing the scalar potentials $q^{\mathrm{S}}, q^{\mathrm{F}}, \zeta^{\mathrm{S}}, \zeta^{\mathrm{F}}$ and vector potentials $\mathbf{U}^{\mathrm{S}}$, $\mathbf{U}^{\mathrm{F}}$, $\mathbf{\Phi}^{\mathrm{S}}$, $\mathbf{\Phi}^{\mathrm{F}}$ through Helmholtz representation of a vector field, we can write

$$\mathbf{u}^{\mathrm{S}} = \nabla q^{\mathrm{S}} + \nabla\times\mathbf{U}^{\mathrm{S}}, \quad \nabla\cdot\mathbf{U}^{\mathrm{S}} = 0, \qquad (6)$$

$$\phi^{\mathrm{S}} = \nabla\zeta^{\mathrm{S}} + \nabla\times\mathbf{\Phi}^{\mathrm{S}}, \quad \nabla\cdot\mathbf{\Phi}^{\mathrm{S}} = 0, \qquad (7)$$

$$\mathbf{u}^{\mathrm{F}} = \nabla q^{\mathrm{F}} + \nabla\times\mathbf{U}^{\mathrm{F}}, \quad \nabla\cdot\mathbf{U}^{\mathrm{F}} = 0, \qquad (8)$$

$$\phi^{\mathrm{F}} = \nabla\zeta^{\mathrm{F}} + \nabla\times\mathbf{\Phi}^{\mathrm{F}}, \quad \nabla\cdot\mathbf{\Phi}^{\mathrm{F}} = 0, \qquad (9)$$

Substituting Eqs. 6–9 into Eqs. 1–5, we obtain the following nine equations, i.e., three coupled equations in $q^{\mathrm{S}}, q^{\mathrm{F}}, T$, two coupled equations in $\zeta^{\mathrm{S}}, \zeta^{\mathrm{F}}$, and four coupled equations in $\mathbf{U}^{\mathrm{S}}, \mathbf{U}^{\mathrm{F}}, \mathbf{\Phi}^{\mathrm{S}}, \mathbf{\Phi}^{\mathrm{F}}$.

$$(c^2{}_{1\mathrm{S}} + c^2{}_{3\mathrm{S}})\nabla^2 q^{\mathrm{S}} - \bar{\xi}^{\mathrm{S}}(\dot{q}^{\mathrm{S}} - \dot{q}^{\mathrm{F}}) - \bar{\beta_1}^{\mathrm{S}}(T + t_1\dot{T}) = \ddot{q}^{\mathrm{S}}, \tag{10}$$

$$(c^2{}_{1\mathrm{F}} + c^2{}_{3\mathrm{F}})\nabla^2 \dot{q}^{\mathrm{F}} + \bar{\xi}^{\mathrm{F}}(\dot{q}^{\mathrm{S}} - \dot{q}^{\mathrm{F}}) + \bar{\beta_1}^{\mathrm{F}}(T + t_1\dot{T}) = \ddot{q}^{\mathrm{F}}, \tag{11}$$

$$\begin{aligned} &\bar{K}\nabla^2 T + \bar{s}\nabla^2(\dot{q}^{\mathrm{S}} - \dot{q}^{\mathrm{F}}) - \bar{\beta}_0\left(1 + t_0\Delta\frac{\partial}{\partial t}\right)\nabla^2\dot{q}^{\mathrm{S}} \\ &\quad - \bar{\delta}\left(1 + t_0\Delta\frac{\partial}{\partial t}\right)\nabla^2\dot{q}^{\mathrm{F}} = \left(1 + t_0\frac{\partial}{\partial t}\right)\dot{T}, \end{aligned} \tag{12}$$

$$(c^2{}_{4\mathrm{S}} + c^2{}_{5\mathrm{S}})\nabla^2 \zeta^{S} - 2\omega^2{}_{0\mathrm{S}}\zeta^{\mathrm{S}} - \bar{\omega_j}^{\mathrm{S}}(\dot{\zeta}^{\mathrm{S}} - \dot{\zeta}^{\mathrm{F}}) = \ddot{\zeta}^{\mathrm{S}}, \tag{13}$$

$$(c^2{}_{4\mathrm{F}} + c^2{}_{5\mathrm{F}})\nabla^2 \dot{\zeta}^{\mathrm{F}} - 2\omega^2{}_{0\mathrm{F}}\dot{\zeta}^{\mathrm{F}} + \bar{\omega_j}^{\mathrm{F}}(\dot{\zeta}^{\mathrm{S}} - \dot{\zeta}^{\mathrm{F}}) = \ddot{\zeta}^{\mathrm{F}}, \tag{14}$$

$$(c^2{}_{2\mathrm{S}} + c^2{}_{3\mathrm{S}})\nabla^2 \mathbf{U}^{\mathrm{S}} + c^2{}_{3\mathrm{S}}\nabla \times \mathbf{\Phi}^{\mathrm{S}} - \bar{\xi}^{\mathrm{S}}(\dot{\mathbf{U}}^{\mathrm{S}} - \dot{\mathbf{U}}^{\mathrm{F}}) = \ddot{\mathbf{U}}^{\mathrm{S}}, \tag{15}$$

$$(c^2{}_{2\mathrm{F}} + c^2{}_{3\mathrm{F}})\nabla^2 \dot{\mathbf{U}}^{\mathrm{F}} + c^2{}_{3\mathrm{F}}\nabla \times \dot{\mathbf{\Phi}}^{\mathrm{F}} + \bar{\xi}^{\mathrm{F}}(\dot{\mathbf{U}}^{\mathrm{S}} - \dot{\mathbf{U}}^{\mathrm{F}}) = \ddot{\mathbf{U}}^{\mathrm{F}}, \tag{16}$$

$$c^2{}_{4\mathrm{S}}\nabla^2 \mathbf{\Phi}^{\mathrm{S}} - 2\omega^2{}_{0\mathrm{S}}\mathbf{\Phi}^{\mathrm{S}} + \omega^2{}_{0\mathrm{S}}\nabla \times \mathbf{U}^{\mathrm{S}} - \bar{\omega_j}^{\mathrm{S}}(\dot{\mathbf{\Phi}}^{\mathrm{S}} - \dot{\mathbf{\Phi}}^{\mathrm{F}}) = \ddot{\mathbf{\Phi}}^{\mathrm{S}}, \tag{17}$$

$$c^2{}_{4\mathrm{F}}\nabla^2 \dot{\mathbf{\Phi}}^{\mathrm{F}} - 2\omega^2{}_{0\mathrm{F}}\dot{\mathbf{\Phi}}^{\mathrm{F}} + \omega^2{}_{0\mathrm{F}}\nabla \times \dot{\mathbf{U}}^{\mathrm{F}} + \bar{\omega_j}^{\mathrm{F}}(\dot{\mathbf{\Phi}}^{\mathrm{S}} - \dot{\mathbf{\Phi}}^{\mathrm{F}}) = \ddot{\mathbf{\Phi}}^{\mathrm{F}}, \tag{18}$$

where $c^2{}_{1\mathrm{S}} = (\lambda^{\mathrm{S}} + 2\mu^{\mathrm{S}})/\rho^{\mathrm{S}}, c^2{}_{2\mathrm{S}} = \mu^{\mathrm{S}}/\rho^{\mathrm{S}}, c^2{}_{3\mathrm{S}} = \kappa^{\mathrm{S}}/\rho^{\mathrm{S}}, c^2{}_{4\mathrm{S}} = \gamma^{\mathrm{S}}/\rho^{\mathrm{S}} j^{\mathrm{S}}$, $c^2{}_{5\mathrm{S}} = (\alpha^{\mathrm{S}} + \beta^{\mathrm{S}})/\rho^{\mathrm{S}} j^{\mathrm{S}}, \omega^2{}_{0\mathrm{S}} = \kappa^{\mathrm{S}}/\rho^{\mathrm{S}} j^{S}, \bar{\xi}^{S} = \xi/\rho^{S}, \bar{\beta_1}^{S} = \beta_1/\rho^{\mathrm{S}}, \bar{\omega_j}^{\mathrm{S}} = \bar{\omega}/\rho^{\mathrm{S}} j^{\mathrm{S}}, c^2{}_{1\mathrm{F}} = (\lambda^{\mathrm{F}} + 2\mu^{\mathrm{F}})/\rho^{\mathrm{F}}, c^2{}_{2\mathrm{F}} = \mu^{\mathrm{F}}/\rho^{\mathrm{F}}, c^2{}_{3\mathrm{F}} = \kappa^{\mathrm{F}}/\rho^{\mathrm{F}}, c^2{}_{4\mathrm{F}} = \gamma^{\mathrm{F}}/\rho^{\mathrm{F}} j^{\mathrm{F}}, c^2{}_{5\mathrm{F}} = (\alpha^{\mathrm{F}} + \beta^{\mathrm{F}})/\rho^{\mathrm{F}} j^{\mathrm{F}}, \omega^2{}_{0\mathrm{F}} = \kappa^{\mathrm{F}}/\rho^{\mathrm{F}} j^{\mathrm{F}}, \bar{\xi}^{\mathrm{F}} = \xi/\rho^{\mathrm{F}}, \bar{\beta_1}^{\mathrm{F}} = \beta_1/\rho^{\mathrm{F}}, \bar{\omega_j}^{\mathrm{F}} = \bar{\omega}/\rho^{\mathrm{F}} j^{\mathrm{F}}, \beta_1 = \frac{s}{T_0} + \beta_0, \bar{\beta}_0 = \beta_0/C_0, \bar{\delta} = \rho\delta/C_0, \bar{K} = K/\rho C_0 T_0$, $\bar{s} = s/\rho C_0$.

## 3 Wave Propagation

In this section, plane wave propagation in an infinite thermally conducting mixture of a micropolar solid and a micropolar viscous fluid is studied. In order to solve Eqs. 10–14, the following form of plane waves propagating in the positive direction of a unit vector **n** is considered:

$$\{q^{\mathrm{S}}, q^{\mathrm{F}}, T, \zeta^{\mathrm{S}}, \zeta^{\mathrm{F}}\} = \{\bar{q}^{\mathrm{S}}, \bar{q}^{\mathrm{F}}, \bar{T}, \bar{\zeta}^{\mathrm{S}}, \bar{\zeta}^{\mathrm{F}}\} \quad \mathtt{exp}[\mathtt{i}k(\mathbf{n}\cdot\mathbf{r} - Vt)], \tag{19}$$

where $\bar{q}^{\mathrm{S}}, \bar{q}^{\mathrm{F}}, \bar{T}, \bar{\zeta}^{\mathrm{S}}, \bar{\zeta}^{\mathrm{F}}$ are the constant complex scalar wave amplitudes, $\mathtt{i} = \sqrt{-1}$, $\mathbf{r}$ is the position vector, $V$ is the phase velocity in the direction of $\mathbf{n}$, $k$ is the wavenumber, and ($\omega = kV$) is the angular frequency. Using the values of $q^{\mathrm{S}}, q^{\mathrm{F}}$, and $T$ from Eq. 19 into Eqs. 10–12, a set of three homogeneous equations in three unknown amplitudes $\bar{q}^{\mathrm{S}}, \bar{q}^{\mathrm{F}}, \bar{T}$ is obtained. Eliminating these three unknowns, the following cubic equation is obtained:

$$\Gamma^3 + A\Gamma^2 + B\Gamma + C = 0, \tag{20}$$

where $\Gamma = V^2$, $A = [\mathrm{i}\omega V^2{}_{1\mathrm{F}}\tau'{}_m - V^2{}_{1\mathrm{S}}\tau'{}_m + i\omega\tau'{}_m(\bar{\xi}^{\mathrm{S}} + \bar{\xi}^{\mathrm{F}})/k^2 - K_\tau + a_2\bar{\beta}_1^{\mathrm{F}}\tau_m + (a_1 + a_2)\bar{\beta}_1^{\mathrm{S}}\tau_m, ]/\tau'{}_m$; $B = [-\mathrm{i}\omega K_\tau\{V^2{}_{1\mathrm{F}} + (\bar{\xi}^{\mathrm{F}}/k^2)\} - V^2{}_{1\mathrm{S}}\{\mathrm{i}\omega V^2{}_{1\mathrm{F}}\tau'{}_m + \mathrm{i}\omega(\bar{\xi}^{\mathrm{F}}/k^2)\tau'{}_m - K_\tau + a_2\bar{\beta}_1^{\mathrm{F}}\tau_m\} + \mathrm{i}\omega(\bar{\xi}^{\mathrm{S}}/k^2)\{\mathrm{i}\omega V^2{}_{1\mathrm{F}}\tau'{}_m - K_\tau - a_1\bar{\beta}_1^{\mathrm{F}}\tau_m\} + \mathrm{i}\omega\bar{\beta}_1^{\mathrm{S}}\tau_m\{a_2V^2{}_{1\mathrm{F}} + a_1V^2{}_{1\mathrm{F}} + a_1(\bar{\xi}^{\mathrm{F}}/k^2)\}]/\tau'{}_m$, $C = [\mathrm{i}\omega V^2{}_{1\mathrm{S}}K_\tau\{V^2{}_{1\mathrm{F}} + (\bar{\xi}^{\mathrm{F}}/k^2)\} + \omega^2K_\tau V^2{}_{1\mathrm{F}}(\bar{\xi}^{\mathrm{S}}/k^2)]/\tau'{}_m$, and, $V^2{}_{1\mathrm{S}} = c^2{}_{1\mathrm{S}} + c^2{}_{3\mathrm{S}}$, $V^2{}_{1\mathrm{F}} = c^2{}_{1\mathrm{F}} + c^2{}_{3\mathrm{F}}$, $a_1 = \bar{s}^* - \bar{\beta}_0$, $a_2 = \bar{s}^* + \bar{\delta}$, $\bar{s}^* = \bar{s}/(1 - \mathrm{i}\omega t_0\Delta)$, $K_\tau = \bar{K}/(t_0\Delta + \frac{\mathrm{i}}{\omega})$, $\tau_m = 1 - \mathrm{i}\omega t_1$, $\tau'{}_m = (1 - \mathrm{i}\omega t_0)/(1 - \mathrm{i}\omega t_0\Delta)$.

The three roots of Eq. 20 correspond to complex speeds of three coupled longitudinal displacement waves in an infinite generalized thermoelastic micropolar mixture of porous media.

Similarly, using the values of $\zeta^{\mathrm{S}}$ and $\zeta^{\mathrm{F}}$ from Eq. 19 into Eqs. 13 and 14, a set of two homogeneous equations in two unknowns $\bar{\zeta}^{\mathrm{S}}, \bar{\zeta}^{\mathrm{F}}$ is obtained. By eliminating these two unknowns, the following quadratic equation is obtained:

$$\left(V^2\right)^2 - GV^2 + H = 0, \tag{21}$$

where $G = V^2{}_{2\mathrm{S}} - \mathrm{i}\omega\{V^2{}_{2\mathrm{F}} + (\omega_j{}^{\mathrm{S}} + \omega_j{}^{\mathrm{F}})/k^2\}$, $H = \mathrm{i}\omega\{\mathrm{i}\omega V^2{}_{2\mathrm{F}}(\omega_j{}^{\mathrm{S}}/k^2) - V^2{}_{2\mathrm{S}}(\omega_j{}^{\mathrm{F}}/k^2) - V^2{}_{2\mathrm{S}}V^2{}_{2\mathrm{F}}\}$, and $V^2{}_{2\mathrm{S}} = c^2{}_{4\mathrm{S}} + c^2{}_{5\mathrm{S}} + 2\omega^2{}_{0\mathrm{S}}/k^2$, $V^2{}_{2\mathrm{F}} = c^2{}_{4\mathrm{F}} + c^2{}_{5\mathrm{F}} + 2\omega^2{}_{0\mathrm{F}}/k^2$.

The two roots of Eq. 21 correspond to complex speeds of two coupled longitudinal microrotational waves.

Equations 15–18 are solved by considering the following form of vector potentials:

$$\{\mathbf{U^S}, \mathbf{U^F}, \mathbf{\Phi^S}, \mathbf{\Phi^F}\} = \{\mathbf{A^S}, \mathbf{A^F}, \mathbf{B^S}, \mathbf{B^F}\} \quad \exp[\mathrm{ik}(\mathbf{n}\cdot\mathbf{r} - \mathrm{Vt})], \tag{22}$$

where $\mathbf{A^S}, \mathbf{A^F}, \mathbf{B^S}, \mathbf{B^F}$ are constant complex vector wave amplitudes. Using Eq. 22 into Eqs. 15–18, the four homogeneous vector equations in four unknowns are obtained as

$$a_{11}\mathbf{A^S} + a_{12}(\mathbf{n}\times\mathbf{B^S}) + a_{13}\mathbf{A^F} = 0, \tag{23}$$

$$a_{21}\mathbf{A^S} + a_{23}\mathbf{A^F} + a_{24}(\mathbf{n}\times\mathbf{B^F}) = 0, \tag{24}$$

$$a_{31}(\mathbf{n}\times\mathbf{A^S}) + a_{32}\mathbf{B^S} + a_{34}\mathbf{B^F} = 0, \tag{25}$$

$$a_{42}\mathbf{B^S} + a_{43}(\mathbf{n}\times\mathbf{A^F}) + a_{44}\mathbf{B^F} = 0, \tag{26}$$

where $a_{11} = k^2(V^2 - c^2{}_{2\mathrm{S}} - c^2{}_{3\mathrm{S}}) + \mathrm{i}\omega\bar{\xi}^{\mathrm{S}}$, $a_{12} = ikc^2{}_{3\mathrm{S}}$, $a_{13} = -\mathrm{i}\omega\bar{\xi}^{\mathrm{S}}$, $a_{21} = -\mathrm{i}\omega\bar{\xi}^{\mathrm{F}}$, $a_{23} = k^2\{V^2 - \mathrm{i}\omega(c^2{}_{2\mathrm{F}} + c^2{}_{3\mathrm{F}})\} + \mathrm{i}\omega\bar{\xi}^{\mathrm{F}}$, $a_{24} = k\omega c^2{}_{3\mathrm{F}}$, $a_{31} = \mathrm{i}k\omega^2{}_{0\mathrm{S}}$, $a_{32} = k^2(V^2 - c^2{}_{4\mathrm{S}} - 2\omega^2{}_{0\mathrm{S}}/k^2) + \mathrm{i}\omega\omega_j{}^{\mathrm{S}}$, $a_{34} = -\mathrm{i}\omega\omega_j{}^{\mathrm{S}}$, $a_{42} = -\mathrm{i}\omega\omega_j{}^{\mathrm{F}}$, $a_{43} = k\omega\omega^2{}_{0\mathrm{F}}$, $a_{44} = k^2(V^2 + \mathrm{i}\omega c^2{}_{4\mathrm{F}} + 2\mathrm{i}\omega\omega^2{}_{0\mathrm{F}}/k^2) + \mathrm{i}\omega\omega_j{}^{\mathrm{F}}$.

Eliminating the vectors $\mathbf{A^S}$, $\mathbf{A^F}$, $\mathbf{B^S}$, $\mathbf{B^F}$, the following relations are obtained:

$$(d_{11} - f_{11})(e_{12} - g_{12}) - (e_{11} - g_{11})(d_{12} - f_{12}) = 0, \tag{27}$$

where $d_{11} = \frac{a_{12}a_{23}}{a_{11}a_{23}-a_{21}a_{13}}$, $d_{12} = -\frac{a_{24}a_{13}}{a_{11}a_{23}-a_{21}a_{13}}$, $e_{11} = \frac{a_{12}a_{21}}{a_{21}a_{13}-a_{11}a_{23}}$, $e_{12} = -\frac{a_{11}a_{24}}{a_{21}a_{13}-a_{11}a_{23}}$, $f_{11} = -\frac{a_{32}}{a_{31}}$, $f_{12} = -\frac{a_{34}}{a_{31}}$, $g_{11} = -\frac{a_{42}}{a_{43}}$, $g_{12} = -\frac{a_{44}}{a_{43}}$.

Equation 27 gives six values of $V^2$, which correspond to six dispersive coupled transverse waves similar to those obtained by Singh and Tomar [24].

## 4 Particular Cases

(i) For the L-S (Lord–Shulman) theory, we put $t_1 = 0$, $\Delta = 1$; therefore, $\tau_m = 1$, $\tau'_m = 1$, $\bar{s}^* = \bar{s}/(1 - \mathrm{i}\omega t_0)$, and $K_\tau = \bar{K}/(t_0 + \frac{i}{\omega})$. Using these values, the three roots of Eq. 20 correspond to the complex speeds of three coupled longitudinal displacement waves in a thermoelastic micropolar mixture of porous media for the case of the Lord–Shulman theory of generalized thermoelasticity.

(ii) For the G-L (Green–Lindsay) theory, we put $\Delta = 0$; therefore, $\tau_m = (1 - \mathrm{i}\omega t_1)$, $\tau'_m = (1 - \mathrm{i}\omega t_0)$, $\bar{s}^* = \bar{s}$, and $K_\tau = \bar{K}/(\mathrm{i}/\omega)$. Using these values, the three roots of Eq. 20 correspond to the complex speeds of three coupled longitudinal displacement waves in a thermoelastic micropolar mixture of porous media for the case of the Green–Lindsay theory of generalized thermoelasticity.

(iii) If we neglect the presence of a fluid, Eq. 20 reduces to

$$\tau'_m (V^2)^2 - \{K_\tau + \tau'_m V^2{}_{1\mathrm{S}} + (\beta_0{}^2/\rho C_0)\tau_m\} V^2 + K_\tau V^2{}_{1\mathrm{S}} = 0, \tag{28}$$

which is a quadratic equation that gives two roots corresponding to the complex speeds of two coupled longitudinal displacement waves in generalized thermoelastic micropolar media for both L-S and G-L theories.

In the absence of fluid, the quadratic Eq. 21 reduces to $V^2 - V^2{}_{2\mathrm{S}} = 0$, which gives the speed of longitudinal microrotational waves in a classical micropolar elastic solid as obtained by Parfitt and Eringen [25]. Similarly, Eq. 27 reduces to

$$a_{12}{}^* a_{31}{}^* + a_{11}{}^* a_{32}{}^* = 0, \tag{29}$$

where $a_{11}{}^* = k^2(V^2 - V^2{}_{2\mathrm{S}})$, $a_{12}{}^* = \mathrm{i}kc^2{}_{3\mathrm{S}}$, $a_{31}{}^* = \mathrm{i}k\omega^2{}_{0\mathrm{S}}$, and $a_{32}{}^* = k^2(V^2 - V^2{}_{4\mathrm{S}})$. Equation 29 is a quadratic equation in $V^2$, and its roots correspond to the speeds of two coupled transverse waves as obtained by Parfitt and Eringen [25].

(iv) If we neglect the thermal effects, the cubic Eq. 20 reduces to the following quadratic equation

$$(V^2)^2 + A^* V^2 + B^* = 0, \tag{30}$$

where $A^* = \mathrm{i}\omega V^2{}_{1\mathrm{F}} - V^2{}_{1\mathrm{S}} + \mathrm{i}\omega(\bar{\xi}^{\mathrm{S}} + \bar{\xi}^{\mathrm{F}})/k^2$, $B^* = -\omega^2 V^2{}_{1\mathrm{F}}(\bar{\xi}^{\mathrm{S}}/k^2) - \mathrm{i}\omega V^2{}_{1\mathrm{S}} V^2{}_{1\mathrm{F}} - \mathrm{i}\omega V^2{}_{1\mathrm{S}}(\bar{\xi}^{\mathrm{F}}/k^2)$. The roots of Eq. 30 correspond to the speeds of two coupled longitudinal displacement waves as obtained by Singh and Tomar [23].

## 5 Numerical Results and Discussion

In this section, the phase speeds ($v$) and the inverse quality factors ($1/Q$) of coupled longitudinal displacement waves are computed using the relation, $V^{-1} = v^{-1} - \mathrm{i}\omega^{-1}Q$. These phase speeds and inverse quality factors are shown graphically with a certain range of the frequency. The following physical constants are considered for numerical computations of phase speeds and inverse quality factors: $\lambda^{\mathrm{S}} = 7.59 \times 10^9\,\mathrm{Pa}$, $\mu^{\mathrm{S}} = 1.89 \times 10^9\,\mathrm{Pa}$, $\kappa^{\mathrm{S}} = 0.0149 \times 10^9\,\mathrm{Pa}$, $\lambda^{\mathrm{F}} = 2.14 \times 10^9\,\mathrm{Pa}$, $\mu^{\mathrm{F}} = 0.45 \times 10^9\,\mathrm{Pa}$, $\kappa^{\mathrm{F}} = 0.0112 \times 10^9\,\mathrm{Pa}$, $\rho^{\mathrm{S}} = 2192\,\mathrm{g \cdot cm^{-3}}$, $\rho^{\mathrm{F}} = 1010\,\mathrm{g \cdot cm^{-3}}$, $C_0 = 0.023\,\mathrm{cal \cdot g^{-1} \cdot {}^{\circ}C^{-1}}$, $K = 0.006\,\mathrm{cal} \cdot \mathrm{cm}^{-1} \cdot s^{-1} \cdot {}^{\circ}\mathrm{C}^{-1}$, $\mathrm{s} = 0.005\,\mathrm{cal}$, $\xi = 0.75\,\mathrm{g \cdot cm^{-3}} \cdot s^{-1}$, $T_0 = 30\,{}^{\circ}\mathrm{C}$, $t_0 = 0.005\,\mathrm{s}$, $t_1 = 0.006\,\mathrm{s}$, $\beta_0 = 0.0005$, $\delta = 0.0001$, and $\mathrm{n} = 0.15$. The densities of solid, fluid, and mixture are related as $\rho = (1-n)\rho^{\mathrm{S}} + n\rho^{\mathrm{F}}$, where $n$ is porosity of the mixture.

The phase speeds and the inverse quality factors of three coupled longitudinal displacement waves are computed numerically for the L-S (Lord–Shulman), the G-L (Green–Lindsay), and the W-T (without thermal effects) cases for the range $0 < \omega \leq 1000\,\mathrm{Hz}$ of frequency. The phase speeds and inverse quality factors for the L-S and the G-L cases are found to be the same up to the first three decimal places. Therefore,

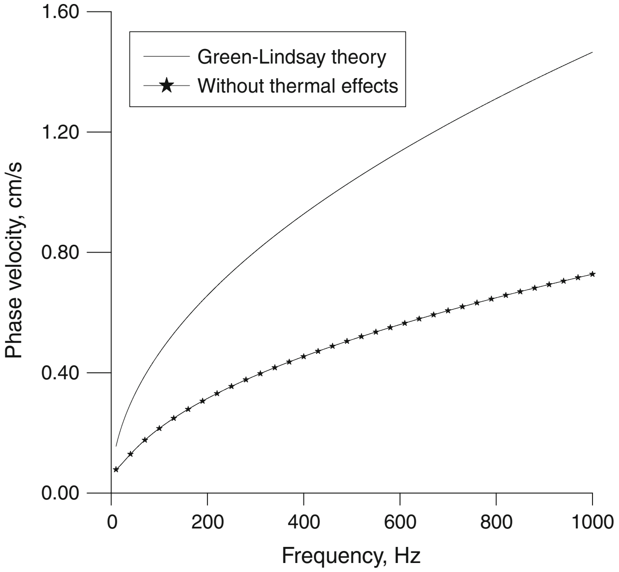

**Fig. 1** Phase velocity of coupled longitudinal wave I

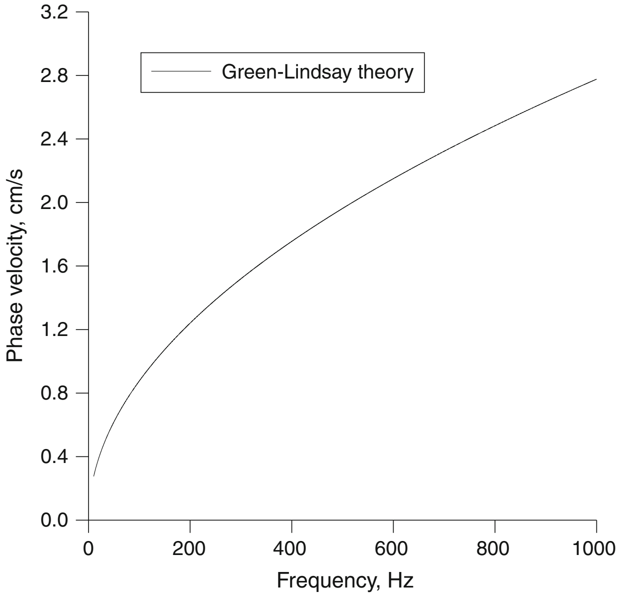

**Fig. 2** Phase velocity of coupled longitudinal wave II

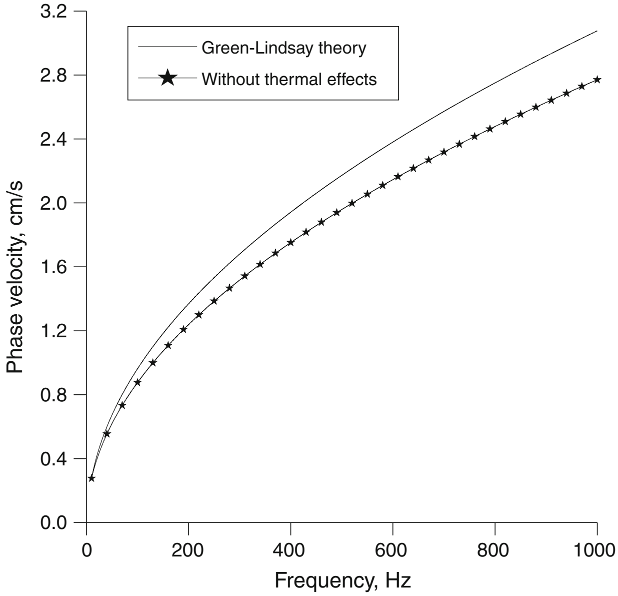

**Fig. 3** Phase velocity of coupled longitudinal wave III

the variations of phase speeds and inverse quality factors are shown graphically as a function of frequency for the G-L and the W-T cases only. The phase speeds of coupled longitudinal displacement waves increase first sharply and then slowly with

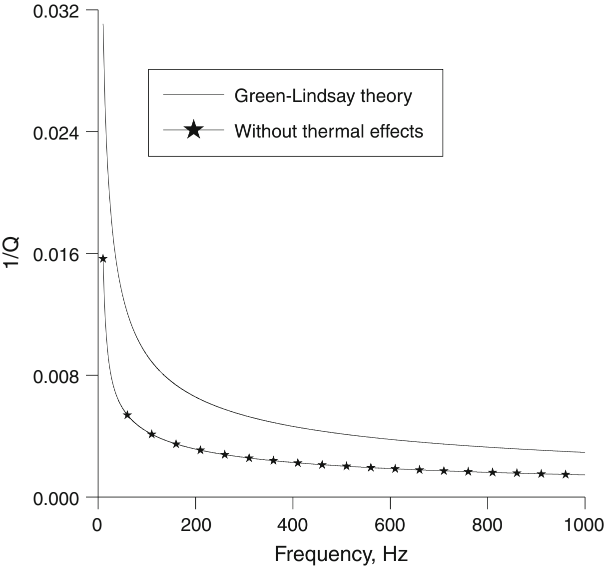

**Fig. 4** Inverse quality factor of coupled longitudinal wave I

the increase in the frequency whereas the inverse quality factors decrease first sharply and then slowly with an increase in the frequency. The phase speed and inverse quality factor of a coupled longitudinal displacement wave I for the G-L and the W-T cases are shown graphically in Figs. 1 and 4. It is observed that the phase speed and inverse quality factor of coupled longitudinal displacement wave I increase in the presence of thermal disturbances.

The phase speed and inverse quality factor of coupled longitudinal displacement wave II for the G-L case are shown graphically as a function of frequency in Figs. 2 and 5. In the absence of thermal disturbances, the curves in these figures will disappear.

The phase speed and inverse quality factor of coupled longitudinal displacement wave III for the G-L and the W-T cases are shown graphically in Figs. 3 and 6. The phase speed and inverse quality factor of coupled longitudinal displacement wave III also increase due to the presence of thermal disturbances.

## 6 Conclusion

The equations governing the generalized thermoelasticity of a mixture of a micropolar elastic solid and a micropolar viscous fluid are solved to show the existence of three coupled longitudinal displacement waves, two coupled longitudinal microrotational waves, and six coupled transverse waves. These plane waves are dispersive in nature. The phase speeds and inverse quality factors of three coupled longitudinal waves are computed against the frequency for the Lord–Shulman, the Green–Lindsay, and the “without thermal effects” cases. The numerical results for the L-S and the G-L cases

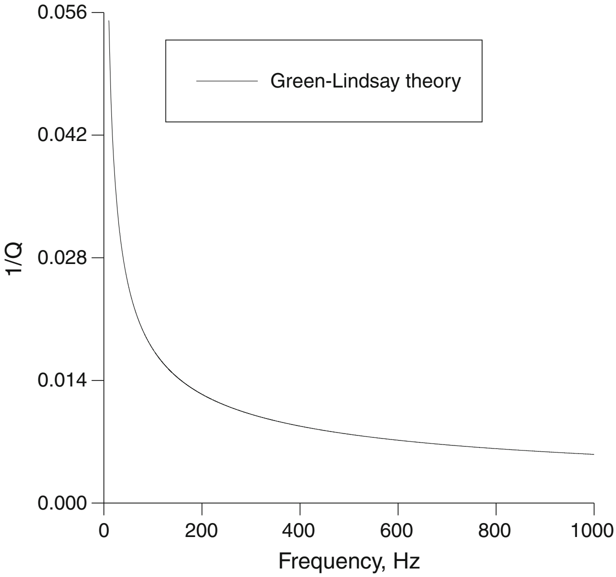

**Fig. 5** Inverse quality factor of coupled longitudinal wave II

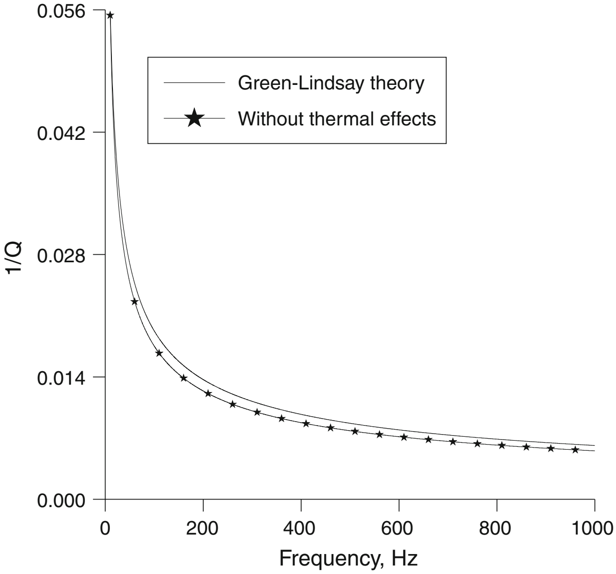

**Fig. 6** Inverse quality factor of coupled longitudinal wave III

are found to be different only after three decimal places. The comparison of numerical results for the L-S and the G-L cases shows the effect of the second relaxation time, which is not significant. The phase speeds and inverse quality factors of three coupled longitudinal waves are plotted against the frequency for the G-L and the W-T cases to show the thermal effects. It is observed that the phase speeds and inverse quality factors of coupled longitudinal waves are affected significantly by the presence of thermal disturbances.

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